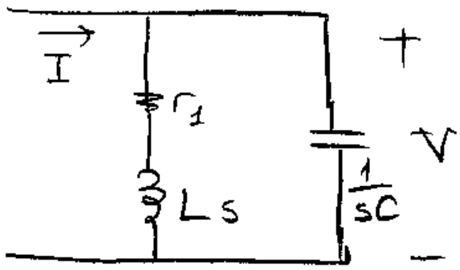
Effect of Internal Resistance of an Inductor

Parallel LC:



$$H(s) = \frac{V(s)}{I(s)} = Z(s)$$

$$Z(s) = \frac{(r_1 + sL)\cdot\frac{1}{sC}}{r_1 + sL + \frac{1}{sC}} = \frac{r_1 + sL}{s^2 LC + s r_1 C + 1}$$

$$\boxed{H(s) = Z(s) = \frac{s\cdot\frac{1}{C} + \frac{r_1}{LC}}{s^2 + s\frac{r_1}{L} + \frac{1}{LC}}}$$

Define: $\omega_a \triangleq \frac{1}{\sqrt{LC}}$: Not the resonance frequency!

$$Q \triangleq \frac{\omega_a}{r_1/L} = \frac{1}{\sqrt{LC}}\cdot\frac{L}{r_1} = \sqrt{\frac{L}{C}}\cdot\frac{1}{r_1}\cdot\frac{\sqrt{C}}{\sqrt{C}}$$

Also NOTE: $\Rightarrow Q = \frac{\sqrt{LC}}{C r_1} = \frac{1}{\omega_a C r_1}$

Resonance Frequency:

At the resonance frequency; $Z(j\omega)$ and $Y(j\omega)$ are purely real.

Use $Z(j\omega)$:

$$Z(j\omega) = \frac{j\frac{\omega}{C} + \frac{r_1}{LC}}{-\omega^2 + \frac{1}{LC} + j\omega\frac{r_1}{L}}$$

$$\left(-\omega^2 + \frac{1}{LC} - j\omega\frac{r_1}{L}\right)$$

$$= \frac{\left(j\frac{\omega}{C} + \frac{r_1}{LC}\right)\left(-\omega^2 + \frac{1}{LC} - j\omega\frac{r_1}{L}\right)}{\left(-\omega^2 + \frac{1}{LC}\right)^2 + \left(\omega\frac{r_1}{L}\right)^2} \longrightarrow D(\omega):\ \text{Real}$$

Note that, at $\omega = \omega_0$; $\mathrm{Im}\{Z(j\omega_0)\} = 0$

$$\mathrm{Im}\{Z(j\omega)\} = \left[\frac{\omega}{C}\left(\frac{1}{LC} - \omega^2\right) - \omega\frac{r_1}{L}\cdot\frac{r_1}{LC}\right]\cdot\frac{1}{D(\omega)}$$

$$\frac{\cancel{\omega_0}}{C}\left(\omega_a^2 - \omega_0^2\right) - \omega_0\frac{r_1^2}{L}.\omega_a^2 = 0$$

$$\omega_a^2 - \omega_0^2 = \frac{C}{L}r_1^2\,\omega_a^2$$

$$\omega_a^2\left(1 - \frac{C}{L}r_1^2\right) = \omega_0^2$$

Recall: $Q = \sqrt{\frac{L}{C}}\cdot\frac{1}{r_1} \Rightarrow r_1^2\cdot\frac{C}{L} = \frac{1}{Q^2}$

$$\Rightarrow \quad \omega_0^2 = \omega_a^2 \left(1 - \frac{1}{Q^2}\right)$$

$$\boxed{\omega_0 = \omega_a \sqrt{1 - \frac{1}{Q^2}}}$$ Resonance frequency

Use $Y(j\omega)$:

$$Y(j\omega) = j\omega C + \frac{1}{r_1 + j\omega L} = \frac{j\omega C r_1 - \omega^2 LC + 1}{r_1 + j\omega L}$$

$$(r_1 - j\omega L)$$

$$= \frac{(1 - \omega^2 LC + j\omega r_1 C)(r_1 - j\omega L)}{\underbrace{r_1^2 + (\omega L)^2}_{D(\omega)}}$$

$$\mathrm{Im}\{Y(j\omega)\} = \frac{1}{D(\omega)} \cdot \left(-\omega L(1 - \omega^2 LC) + \omega r_1 C r_1\right)$$

$$= \frac{1}{D(\omega)} \left(-\omega L + \omega^3 L^2 C + \omega C r_1^2\right)$$

$$\mathrm{Im}\{Y(j\omega_0)\} = 0$$

$$\Rightarrow \quad \omega_0(\omega_0^2 L^2 C + C r_1^2 - L) = 0$$

$$\omega_0^2 = \frac{L - C r_1^2}{L^2 C} = \underbrace{\frac{1}{LC}}_{\omega_a^2} - \underbrace{\frac{r_1^2}{L^2}}_{\omega_a^2 / Q^2}$$

$$\Rightarrow \boxed{\omega_0^2 = \omega_a^2 \left(1 - \frac{1}{Q^2}\right)}$$

$$Z(j\omega_0) = \frac{1}{D(\omega_0)} \cdot \left[\underbrace{r_1\, \omega_a^2 \left(\omega_a^2 - \omega_0^2\right) + \omega_0^2\, \omega_a^2}_{r_1 \cdot \omega_a^4} \right]$$

Let us focus on $D(\omega_0)$:

$$D(\omega_0) = \left(\omega_a^2 - \omega_0^2\right)^2 + \omega_0^2 \underbrace{\frac{r_1^2}{L^2}}_{\frac{\omega_a^2}{Q^2}}$$

$$= \omega_a^4 - 2\,\omega_a^2 \omega_0^2 + \omega_0^4 + \omega_0^2\, \omega_a^2 \cdot \frac{1}{Q^2}$$

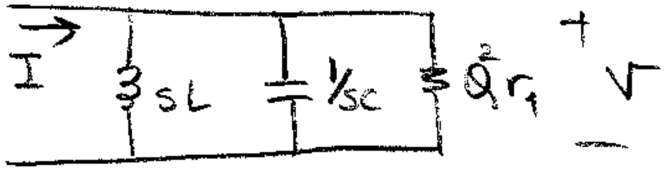
$$= \omega_a^4 \left[1 - 2\, \frac{\omega_0^2}{\omega_a^2} + \frac{\omega_0^4}{\omega_a^4} + \frac{\omega_0^2}{\omega_a^2} \cdot \frac{1}{Q^2} \right]$$

$$\Rightarrow \quad Z(j\omega_0) = \frac{r_1}{1 - 2\left(\frac{\omega_0}{\omega_a}\right)^2 + \left(\frac{\omega_0}{\omega_a}\right)^4 + \frac{1}{Q^2}\left(\frac{\omega_0}{\omega_a}\right)^2}$$

If $\omega_0 \approx \omega_a$ (High Q, $Q \geq 10$)

$$\Rightarrow \quad \boxed{Z(j\omega_0) \approx Q^2 r_1}$$

$\Rightarrow$ High Q equivalent circuit:



Let us rewiret $Z(s)$ in terms of Q and ω_a:

$$Z(s) = \frac{\frac{r_1}{LC} + \frac{s}{C}}{s^2 + s\frac{r_1}{L} + \frac{1}{LC}} = \frac{\frac{s}{C} + r_1\omega_a^2}{s^2 + s\frac{\omega_a}{Q} + \omega_a^2}$$

$$= \frac{\cancel{\omega_a^2} r_1 \left(s \cdot \frac{1}{\omega_a^2 r_1 C} + 1 \right)}{\cancel{\omega_a^2} \left(\frac{s^2}{\omega_a^2} + s \frac{1}{Q\omega_a} + 1 \right)}$$

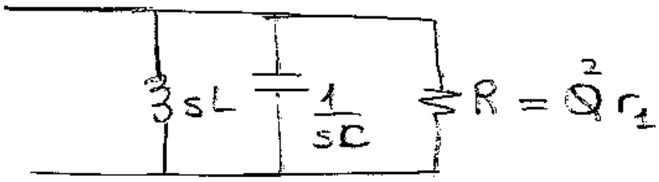
Recall:

$$\frac{1}{\omega_a r_1 C} = Q$$

$$= \frac{r_1 \left(\frac{s}{\omega_a} \cdot Q + 1 \right)}{\left(\frac{s}{\omega_a}\right)^2 + \left(\frac{s}{\omega_a}\right) \cdot \frac{1}{Q} + 1}$$

$$= Q^2 r_1 \cdot \frac{\frac{1}{Q}\left(\frac{s}{\omega_a} + \frac{1}{Q} \right)}{\left(\frac{s}{\omega_a}\right)^2 + \left(\frac{s}{\omega_a}\right) \cdot \frac{1}{Q} + 1}$$

Now, find $Z_{RLC}(s)$ for the approximate RCL:



$$Z_{RLC}(s) = \frac{1}{\frac{1}{sL} + sC + \frac{1}{R}} = \frac{sLR}{R + s^2RLC + sL}$$

$R \qquad sLR \qquad sL$

$$Z_{RLC}(s) = \frac{\frac{1}{C}\cdot s}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$

$$\frac{1}{LC} \longrightarrow \omega_0^2$$

$$2\gamma\omega_0 = BW$$

$$Q^{new} = \frac{\omega_0}{BW} = \omega_0 RC$$

If $\omega_a \approx \omega_0$; $R \approx Q^2 r_1$ where $Q = \frac{\omega_a}{r_1/L} = \frac{1}{\omega_a r_1 C}$

$$Q^{new} = \omega_0 Q^2 r_1 C = \omega_0 \cdot \frac{1}{\omega_a^2 r_1^2 C^2} \cdot r_1 C$$

$$\approx \frac{1}{\omega_a r_1 C} \Rightarrow \boxed{Q^{new} \approx Q}$$

Also note;

$$Z_{RLC}(s) = \frac{\frac{1}{C}\, s}{s^2 + s\frac{1}{RC} + \frac{1}{LC}} \; ;$$

use $\omega_o \approx \omega_a$

$$Q = \frac{1}{\omega_a r_1 C} \approx \frac{1}{\omega_o r_1 C}$$

$$R \approx Q^2 r_1$$

$$= \frac{\frac{1}{C}\, s}{s^2 + s\frac{1}{Q^2 r_1 C} + \omega_o^2}$$

$$\frac{1}{Q.Q r_1 C} = \frac{\omega_o}{Q}$$

$$= \frac{\frac{1}{C} \cdot s}{\omega_o^2\left(\frac{s^2}{\omega_o^2} + \frac{s}{\omega_o}\cdot\frac{1}{Q} + 1\right)}$$

$$\rightarrow \omega_o^2 = \frac{1}{Q^2 r_1^2 C^2}$$

$$= \frac{Q^2 r_1 \, r_1 \, C^2 \cdot \frac{1}{C} \cdot s}{\frac{s^2}{\omega_o^2} + \frac{s}{\omega_o}\cdot\frac{1}{Q} + 1}$$

$$r_1 C = \frac{1}{\omega_o Q}$$

$$\boxed{Z_{RLC}(s) = \frac{Q^2 r_1 \cdot \frac{1}{Q}\frac{s}{\omega_o}}{\left(\frac{s}{\omega_o}\right)^2 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + 1}}$$

$\longleftarrow$ of the RLC

Compare with the true $Z(s)$:

can be neglected for large Q.

$$Z(s) = \frac{Q^2 r_1 \cdot \frac{1}{Q}\left(\frac{s}{\omega_a} + \frac{1}{Q}\right)}{\left(\frac{s}{\omega_a}\right)^2 + \frac{1}{Q}\left(\frac{s}{\omega_a}\right) + 1}$$

If Q is large; $\omega_o \approx \omega_a$; $Z_{RLC}(s) \approx Z(s)$

* The two circuits have approximately
  the same Q,
  the same input impedance
  the same resonance frequencies.

Circuit Interpretation; sketch of $|Z(j\omega)|$:

At $\omega = 0$; L is short; C is open;

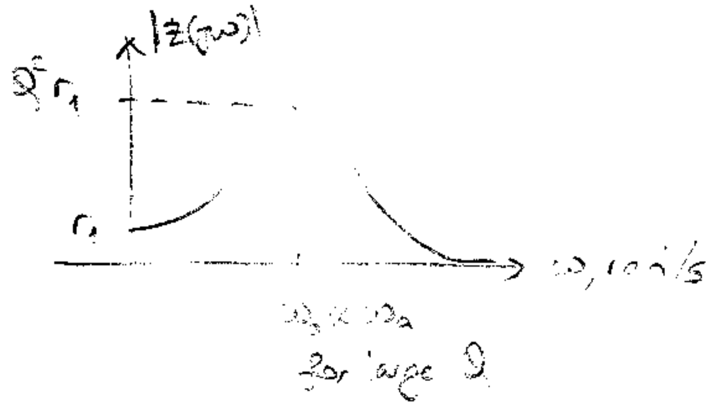
$$|Z(j\omega)| = r_1$$

At $\omega \to \infty$, L is open, C is short;

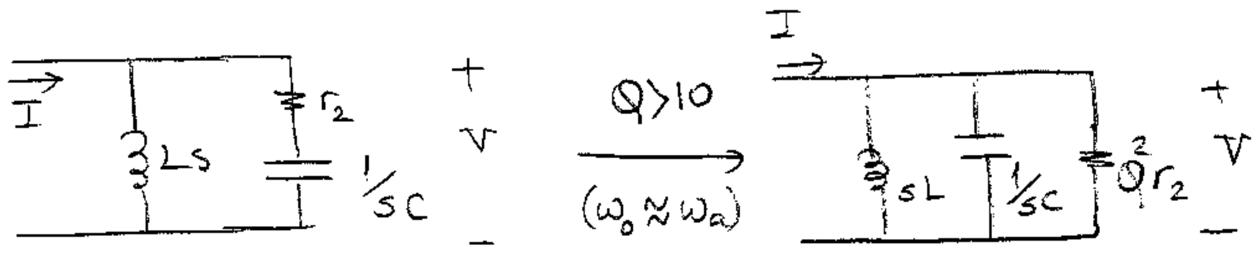
$$|Z(j\omega)| = 0$$

At $\omega = \omega_o$ ($\approx \omega_a$ for large Q)

$$|Z(j\omega)| \approx Q^2 r_1 \quad \text{where } Q \cong \frac{\omega_o}{r_1/L} = \frac{1}{\omega_o r_1 C}$$



Similarly;



$$Q \triangleq \frac{1}{\omega_a r_2 C} = \frac{\omega_a}{r_2/L} \quad ;$$

$$\omega_a \triangleq \frac{1}{\sqrt{LC}}$$

Show that the resonance frequency is:

$$\omega_o = \frac{\omega_a}{\sqrt{1 - 1/Q^2}}$$